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Properties of some chaotic billiards with time-dependent boundaries

A Loskutov, A B Ryabov and L G Akinshin
Physics Faculty, Moscow State University, Moscow 119899, Russia

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Abstract. A dispersing billiard (Lorentz gas) and focusing billiards (in the form of a stadium) with time-dependent boundaries are considered. The problem of particle acceleration in such billiards is studied. For the Lorentz gas two cases of the time dependence are investigated: stochastic perturbations of the boundary and its periodic oscillations. Two types of focusing billiards with periodically forced boundaries are explored: a stadium with strong chaotic properties and a near-rectangle stadium. It is shown that in all cases billiard particles can reach unbounded velocities. Average velocities of the particle ensemble as functions of time and the number of collisions are obtained.

1. Introduction

A billiard is a dynamical system corresponding to the free motion of a point particle (billiard ball) in some manifold Q with a piecewise smooth boundary ∂Q. Reaching the boundary, the particle is reflected from it elastically. This means that the billiard particle moves along geodesic lines with a constant velocity. In this paper we consider billiards in a Euclidean plane. In this case the angle of incidence of the particle is always equal to the angle of reflection.

Suppose that the billiard boundary consists of a finite number of quite smooth components ∂Q_i, $i = 1, 2, \ldots, k$. Each of these components can be supplied by the field of unit internal normals $n(q)$, where a point $q \in \partial Q$. Thus one can define a curvature $\kappa(q)$ for every boundary component ∂Q_i. If, in all points $q \in \partial Q_i$, $\kappa(q) > 0$ then components ∂Q_i are said to be dispersing ones. If $\kappa(q) = 0$ and $\kappa(q) < 0$ then components ∂Q_i are called neutral and focusing components, respectively. Usually the union of dispersing, focusing and neutral components of ∂Q are denoted by ∂Q^+, ∂Q^- and ∂Q^0, respectively.

If the set ∂Q is not perturbed with time then such billiard systems are said to be billiards with a fixed (constant) boundary. In the case of $\partial Q = \partial Q(t)$ the corresponding billiards are called billiards with time-dependent boundaries. Planar billiards with the fixed boundary have been widely studied (see [1–7] and references therein). However, the articles devoted to the systems related to time-dependent billiards are scantily explored (see, e.g., [7–11]). For the most part, investigations of classical time-dependent billiards concerned two main questions: descriptions of their statistical properties and the study of trajectories for which the particle velocity grows indefinitely. The last problem is related to the unbounded increase of energy in periodically forced Hamiltonian systems. It stems from the question concerning the origin of high-energy cosmic particles [12] and known as Fermi acceleration.

To explain the Fermi acceleration a number of models have been proposed (see, e.g., [13–21]). Further development of this problem led to the study of systems of a billiard

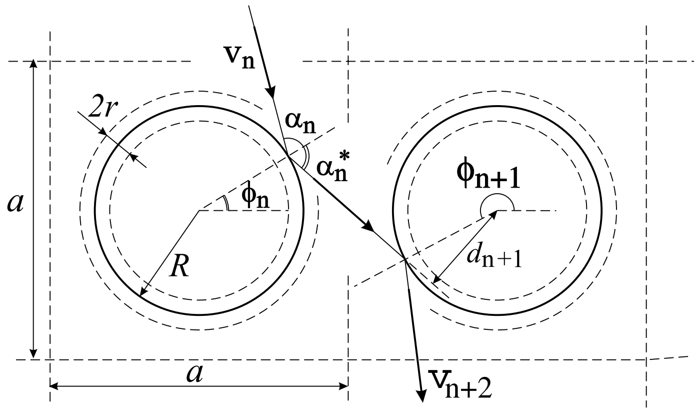

Figure 1. Configuration of Lorentz gas model. The scatterers (circles of radius R) are located at sites of a lattice with period a.

type [8, 10, 11, 22]. In particular, in [8] numerical simulations of the elliptic billiard with time-dependent boundaries have been performed. It was found that in such a system the velocity of the billiard particle is bounded. A process of relaxation to the equilibrium state in a time-dependent billiard system with damping has been considered in [10]. For the example of the Fermi–Ulam model the analogous problem was investigated in [22]. Some other properties of time-dependent billiards were studied in [23, 24].

In this paper we consider the problem of Fermi acceleration in time-dependent billiards of the two following types: in the Lorentz gas with the open horizon and in the billiard in the form of a stadium. Dependence of the mean velocity of the particle ensemble on time and the number of collisions is studied. A more detailed description of some properties of the time-dependent Lorentz gas is given in [25].

2. Lorentz gas

Consider some domain Q with a piecewise smooth boundary ∂Q. A system consisting of dispersing ∂Q_i^+ components of the boundary ∂Q is said to be a dispersing billiard [1, 4]. One of the types of such billiards is a system defined in an unbounded domain D containing a set of heavy discs B_i (scatterers) with boundaries ∂Q_i and radius R embedded at sites of an infinite lattice with period a (see figure 1). If B_i are fixed, the billiard in $Q = D \setminus \bigcup_{i=1}^{r} B_i$ is called a regular *Lorentz gas*. For the case of $\partial Q = \text{const}$ such a billiard has been intensively studied (see, e.g., [3, 4, 6, 26] and references therein).

The ratio $(a/R)^2$ is the fundamental parameter for the Lorentz gas. Depending on this value one can define the Lorentz gas with a bounded horizon ($(a/R)^2 < 4$), with an open horizon ($4 < (a/R)^2 < 8$), and with an infinite horizon ($(a/R)^2 > 8$). In the first case, the particle motion is bounded by a single lattice cell. In the second and third cases the particle can attain each part of the domain Q. For the infinite horizon the statistical properties of a billiard are changed, and there is an algebraic correlation decay with time. But in any case, for billiard maps an exponential decay of correlations takes place. For example, for the velocity autocorrelation function $E(m) = \langle v_n v_{n+m} \rangle$ the following expression holds: $E(m) \leqslant A \exp(-km^\gamma)$, $1/2 \leqslant \gamma \leqslant 1$ [3, 4, 27–29].

It is customary to define a mean free path as follows:

$$l = \int_{\Omega} l(x)\,\mathrm{d}\nu\,(x)$$

where $\Omega = \{(q, v) : q \in \partial Q \text{ and } (v \cdot n(q)) \geqslant 0\}$, $n(q)$ is an internal unit normal to the boundary and v is a velocity vector. The measure $\mathrm{d}\nu$ is defined by $\mathrm{d}\nu\,(x) = c_\nu(v \cdot n(q))\,\mathrm{d}q\,\mathrm{d}v$, where $\mathrm{d}q$ is the Lebesgue measure in ∂Q and c_ν is a normalizing factor.

For any billiard table, depending on its geometric parameters, the mean free path can be obtained as follows (see [28]):

$$l = \frac{|Q||S^{k-1}|}{|\partial Q||B^{k-1}|}$$

where $|S^{k-1}| = 2\pi^{k/2}/\Gamma(k/2)$ is the $(k-1)$-dimensional volume of the unit sphere in $\mathbb{R}^k$, $\Gamma(x)$ is the gamma function and $|B^{k-1}| = |S^{k-2}|/(k-1)$ is the volume of the unit ball in $\mathbb{R}^k$. In particular, for planar billiards $l = \pi|Q|/|\partial Q|$, and for the Lorentz gas with an open horizon $l = (a^2 - \pi R^2)/2R$ [29].

Suppose that radii of all scatterers are perturbed in accordance with a certain law, i.e. all components ∂Q_i make quite small oscillations in the normal direction. We will consider two different cases: periodic (and phase-sychronized) oscillations and stochastic perturbations of the scatterer radii.

To describe the dynamics of the Lorentz gas it is necessary to get the map $(\alpha_n, \phi_n) \to (\alpha_{n+1}, \phi_{n+1})$ (see figure 1) which transform the variables (α, ϕ) in the moment of the nth collision of the particle with ∂Q to their values in the moment of the $(n+1)$th collision. Obviously,

$$\phi_n + \alpha_n^* + \pi = \phi_{n+1} + \alpha_{n+1} \qquad (\mathrm{mod}\ 2\pi) \tag{1}$$

and $\alpha_n^* = -\alpha_n$. Moreover, it is not hard to see that

$$d_{n+1} = a[p\sin(\phi_n + \alpha_n^*) - q\cos(\phi_n + \alpha_n^*)] - R\sin\alpha_n^*. \tag{2}$$

The parameter p is assumed to be positive if the billiard ball moves to the right and negative otherwise. Similarly, q is positive if the ball moves upwards and negative if it moves downwards. Geometrically, p and q are a number of cells through which the ball travels in the horizontal and the vertical, respectively. Thus, the values p and q are defined from the scattering condition as a minimal in absolute value integers for which $|d_{n+1}| \leqslant R$ holds. Now one can find that

$$\alpha_{n+1} = \arcsin\left(\frac{d_{n+1}}{R}\right). \tag{3}$$

The Jacobian of the map (1)–(3) is $\partial(\phi_{n+1}, \alpha_{n+1})/\partial(\phi_n, \alpha_n) = \cos\alpha_n/\cos\alpha_{n+1}$. Thus, the phase volume $\cos\alpha\,\mathrm{d}\alpha\,\mathrm{d}\phi$ is preserving. Therefore, in the ergodic case the distribution of the angle α_n is the following:

$$\rho_\alpha(\alpha) = \tfrac{1}{2}\cos\alpha \tag{4}$$

where $1/2$ is the normalizing factor.

Assume that the scatterer boundaries ∂Q^+ are changed in such a way that $R(t) = R + r(t)$ (see figure 1), where $\max|r(t)| \ll R$. Then the boundary velocity $u(t)$ depends on time as follows: $u(t) = \dot{r}(t)$, for example, $u(t) = u_0\cos\omega t$. In this case, except for variables α and ϕ, it is necessary to introduce two new variables: the particle velocity $\boldsymbol{v}$ and impact time t. Taking into account that, in the process of scattering only the radial component of $\boldsymbol{v}$ is changed, for the absolute value of the particle velocity we get

$$v_{n+1} = \sqrt{v_n^2 - 4u_n v_n\cos\alpha_n + 4u_n^2} \tag{5}$$

where $u_n \equiv u_0 \cos \omega t_n$ is the boundary velocity at the moment of the nth collision. In turn, $\alpha_n^* = -\arcsin[\frac{v_n}{v_{n+1}} \sin \alpha_n]$. Now we can write a map for the impact time:

$$\begin{aligned} t_{n+1} &= t_n + \frac{l_{n+1}}{v_{n+1}} \\ l_{n+1} &= \sqrt{(R(\cos\phi_{n+1} - \cos\phi_n) - pa)^2 + (R(\sin\phi_{n+1} - \sin\phi_n) - qa)^2}. \end{aligned} \tag{6}$$

Here l_n is the free path. Strictly speaking, to take into account the boundary oscillations in (2), (3) and (6) we should substitute the value $R(t_{n+1}) = R + r(t_n)$, instead of R. As a result, the billiard map will be implicit. However, because $r \ll R$ (see above) then the geometric changes of the billiard can be neglected, and we have $R = \text{const}$.

3. Fermi acceleration in the Lorentz gas

As a result of collisions with the scatterers the particle velocity is always changing. In this section we consider an ensemble of particles and find their velocity distribution and average velocity as a function of time t and the number of collisions n. It is quite clear that the number of collisions and time are not in proportion to each other because a 'fast' particle undergoes more collisions than a 'slow' one during the same period of time.

3.1. Stochastic perturbation of the scatterer boundaries

Suppose that the boundary velocity of scatterers at which the nth collision takes place is

$$u_n = u_0 \cos\phi_n \tag{7}$$

where u_0 is an amplitude and $\{\phi_n\}$ is a set of uncorrelated random values equidistributed over the interval $[0, 2\pi)$. Let us find the average velocity of a particle ensemble as a function of time t and number of collisions n. If the velocities of particles are low ($v \ll u_0$) then in (5) the last term plays the main role. Thus, $v_{n+1} \approx 2|u(t_n)|$. When the boundary oscillations are defined by (7) then

$$\langle v_{n+1}\rangle \approx 2\langle |u(t_n)|\rangle_t = 4\frac{u_0}{\pi}.$$

Therefore, after the first collision the average velocity becomes larger than u_0.

Now let us find the change in the velocity at $v \gg u_0$. Expanding (5) into a Taylor series by the parameter u/v, we get

$$\Delta v_n = v_{n+1} - v_n = -2u_n \cos\alpha_n + 2\frac{u_n^2}{v_n}\sin^2\alpha_n + v_n \mathrm{O}\left(\left(\frac{u_n}{v_n}\right)^3\right) \tag{8}$$

where u_n is the scatterer boundary velocity at the nth collision. Using (4) and the condition of uniform distribution in phase oscillation at the moment of collision, we can obtain $\langle \Delta v_n\rangle$ and $\langle(\Delta v_n)^2\rangle$:

$$\begin{aligned} \mu_s &\equiv \langle \Delta v_n\rangle = \frac{M_s}{v} \\ \sigma_s^2 &\equiv \langle(\Delta v_n)^2\rangle = \tfrac{4}{3}u_0^2. \end{aligned} \tag{9}$$

Here we have introduced the parameter $M_s \equiv u_0^2/3$; the subscript s denotes the stochastic case. Clearly, after averaging, only the second term on the right of (8) gives a nonzero contribution. However, for calculations of the dispersion it is sufficient to consider the contribution of the first term.

If the number of collisions n is sufficiently large then, from the first equation in (9), one can pass to the following differential equation:

$$\frac{\partial v(n)}{\partial n} = \frac{M_s}{v(n)}. \tag{10}$$

Its solution with the initial condition $v(0) = v_0$ gives the most probable velocity as a function of the number of collisions:

$$v(n) = \sqrt{2M_s n + v_0^2}. \tag{11}$$

Because the velocity of particles can be presented as a sum of independent random quantities Δv_n with known average value and dispersion, then from the Lyapunov central limit theorem we obtain that the distribution function of the random value $v_n = v_0 + \sum_{i=1}^{n} \Delta v_i$ tends to a normal distribution $N(v(n), n\sigma_s^2)$. Thus, the velocity distribution has the form of a spreading Gaussian. The maximum of this distribution is given by the most probable velocity $v(n)$ which increases as $\sim\sqrt{n}$.

The obtained results are true only for *high* velocities ($v \gg u_0$). To describe the distribution at lower velocities let us introduce an additional condition concerning the absence of the particle flow into the region of negative velocities: $v\partial\rho/\partial v|_{v=0} = 0$. As is well known, the Gaussian distribution which satisfies this condition has the following form:

$$\rho(v, n) = \frac{1}{\sigma_s\sqrt{2\pi n}} \left[\exp\left(-\frac{(v - v(n))^2}{2\sigma_s^2 n}\right) + \exp\left(-\frac{(v + v(n))^2}{2\sigma_s^2 n}\right)\right]. \tag{12}$$

This allows us to find the average velocity in the particle ensemble as a function of the number of collisions:

$$V(n) = \sigma_s\sqrt{\frac{2n}{\pi}} \exp\left(-\frac{v(n)^2}{2\sigma_s^2 n}\right) + v(n)\Phi\left(\frac{v(n)}{\sigma_s\sqrt{2n}}\right) \tag{13}$$

where $\Phi(x) = 2/\sqrt{\pi}\int_0^x \exp(-x^2)\,\mathrm{d}x$ is the known error integral. Expanding (13) into a series, we get

$$V(n) = c\sqrt{n} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right) \tag{14}$$

where $c = \sqrt{2}(\sigma_s \mathrm{e}^{-M_s/\sigma_s^2}/\sqrt{\pi} + \Phi(\sqrt{M_s}/\sigma_s)\sqrt{M_s}) \approx 1.143u_0$.

Thus, expressions (12) and (14) describe the dependence of the velocity distribution and the average velocity of the particle ensemble *on the number of collisions*.

To find the particle velocity as a function of *time*, let us consider the Fokker–Planck equation:

$$\frac{\partial\rho(v, t)}{\partial t} = -\frac{\partial}{\partial v}(A\rho(v, t)) + \frac{1}{2}\frac{\partial^2}{\partial v^2}(B\rho(v, t))$$

where the coefficients A and B have the form:

$$A \equiv \left\langle\frac{\Delta v}{\tau}\right\rangle = \frac{M_s}{l}$$

$$B \equiv \left\langle\frac{\Delta v^2}{\tau}\right\rangle = \frac{\sigma_s^2 v}{l}$$

where $\tau = l/v$ is the mean time between collisions, l is the mean free path, Δv and Δv^2 are defined by (9). Then we obtain

$$\frac{\partial\rho(v, t)}{\partial t} = -\frac{M_s}{l}\frac{\partial}{\partial v}\rho(v, t) + \frac{1}{2}\frac{\sigma_s^2}{l}\frac{\partial^2}{\partial v^2}(v\rho(v, t)). \tag{15}$$

If parameters M_s and σ_s are given in accordance with (9), then at $v \gg v_0$ and large times the solution of this equation tends to $\rho(v,t) = \exp(-\frac{v}{2tA})/\sqrt{2tA\pi v}$. In this case we find the average velocity of particles as a function of time:

$$V(t) = \frac{M_s}{l}t + v_0 = \frac{1}{3}\frac{u_0^2}{l}t + v_0. \tag{16}$$

Thus, in the system investigated the Fermi acceleration is observed, and for the large enough particle velocity the average velocity grows as a linear function of time.

3.2. Periodically perturbed boundaries of scatterers

To analyse the change in the particle velocity in dispersing billiards with periodically perturbed boundaries, the following simplified approach can be used. Consider an approximate map for the velocity increment (8). Because correlations of the values α_n decay exponentially (this fact is defined by the billiard geometry) then we get, using (4),

$$\langle \Delta v \rangle_\alpha = -\frac{\pi}{2}u_0 \cos\omega t_n + \frac{u_0^2 \cos\omega t_n}{v_n}. \tag{17}$$

Obviously, during the oscillation period the first summand brings the largest contribution. Therefore, to find correlations in the first approximation it is sufficient to take into account only this term, and the second term can be neglected. On the other hand, inclusion of correlation corrections in the second component generate terms of higher order of smallness than its average. Thus, one can omit correlation effects in the second component. For this reason, it is admissible to calculate independently two values: $\langle \Delta v \rangle = \langle \Delta v \rangle_I + \langle \Delta v \rangle_{II}$, where $\langle \Delta v \rangle_{II} = u_0^2/(3v)$ which is identical to μ_s in the stochastic case (see (9)), and $\langle \Delta v \rangle_I$ is the correction caused by correlations. Rejecting the second summand in (17) we find the following map for the estimation of $\langle \Delta v \rangle_I$:

$$\begin{aligned} v_{n+1} &= v_n + \gamma \cos\theta_n \\ \theta_{n+1} &= \theta_n + \frac{l_{n+1}\omega}{v_{n+1}} \end{aligned} \tag{18}$$

where $\gamma = -\pi u_0/2$ and $\theta_n \equiv \omega t_n$. This map corresponds to the known Ulam map (see [14–21]), but here the free path l_n is a random parameter distributed over a certain interval. Detailed investigations of the map (18) in the context of time-dependent dispersing billiards are described in [25].

Suppose that the particle velocity is so high that its change after n collisions can be neglected. To satisfy this condition it is sufficient to choose in an appropriate way v and u_0. Let us find correlations of the velocity increments Δv_m and Δv_{m+n} (see (17)) for $n \to \infty$. Taking into account in the first approximation only the first terms, we have

$$G(n) \equiv \langle \Delta v_m \Delta v_{m+n} \rangle = u_0^2 \frac{\pi^2}{4} \langle \cos\omega t_m \cos\omega t_{m+n} \rangle.$$

Then $\langle \cos\omega t_m \cos\omega t_{m+n} \rangle = \langle \cos\omega t_m \cos\omega(t_m + S_n) \rangle$, $S_n \equiv \sum_{i=1}^n \tau_{m+i}$, $\tau_i = t_i - t_{i-1}$. Obviously, one can write that $S_n = \sum_{i=1}^n (l + \Delta l_i)/v$, where Δl_i is the deviation from the mean free path under the ith-particle scattering event. Because S_n is the sum of independent random values then its distribution at large n tends to the normal distribution $N(nl, n\sigma_l^2)$, where σ_l^2 is the dispersion of the free path.

Averaging over S_n we obtain the following expression for the correlation function of the velocity increments:

$$G(n) \simeq \frac{\pi^2}{8} u_0^2 \cos(\omega n \tau) \exp\left(-\frac{n}{N}\right) \tag{19}$$

where ω is the frequency of scatterer oscillations and $N = v^2/(\omega^2\sigma_l^2)$. Thus, correlations between sequential changes in the particle velocity are stronger the higher the velocity, and the number of collisions after which correlations decay in e times increases proportionally to v^2.

To estimate the dispersion in the first approximation, let us consider the velocity change after two sequential collisions with the boundary. In this analysis we assume that correlations between three or more velocity changes are negligibly small. The correlator of sequential velocity increments can be estimated as follows:

$$\langle \Delta v_n \Delta v_{n+1} \rangle = u_0^2 \frac{\pi^2}{4} \langle \cos^2 \omega t_n (1 - \mathrm{O}(\tau^2)) \rangle = u_0^2 \frac{\pi^2}{8} + \mathrm{O}\left(\frac{u_0^2}{v^2}\right).$$

Then taking into account (9), we get

$$\sigma_r^2 = \frac{\langle (\Delta v_n + \Delta v_{n+1})^2 \rangle}{2} \approx \left(\frac{4}{3} + \frac{\pi^2}{8}\right) u_0^2. \tag{20}$$

Thus, obtained estimates show that the particle acceleration should be observed in chaotic billiards with periodically perturbed boundaries.

3.3. Numerical simulations

Consider the Lorentz gas model with the following parameters: the amplitude of the boundary oscillation of scatterers $u_0 = 0.01$, the scatterer radii $R = 0.4$, the distance between the scatterer centres $a = 1$, the frequency of boundary oscillations $\omega = 1$, the initial velocity $v_0 = 1$. Thus, the analytical value of the free path is $l = 0.621\,6815$. Numerical estimations of l and its dispersion σ_l^2 for this specific configuration of the Lorentz gas yield $l = 0.621\,63 \pm 0.000\,03$, $\sigma_l^2 = 0.657 \pm 0.001$.

Numerical realizations of particle trajectories were different from each other in the initial values of α and ϕ which were chosen in a random way. Two qualitative different cases were considered: stochastic oscillations of scatterer boundaries with equidistributed phases and periodic oscillations. In both cases, the particle dynamics was determined by the map described in section 2. The oscillation velocity of the boundary was defined as follows: $u_n = u_0 \cos \phi_n$, where ϕ_n is a uniformly distributed random value over the interval $[0, 2\pi)$ (stochastic case), and $u_n = u_0 \cos \omega t_n$, where t_n is the instant of the nth collision with the boundary (periodic oscillations). For each case 5000 realizations of the trajectory of the billiard particle have been constructed. The corresponding averaged dependencies of the particle velocity on the number of collisions and time are shown in figures 2 and 3, respectively.

Figure 2 shows the averaged velocity of the particle ensemble as a function of the number n of collisions during 10^6 iterations (dotted curves). Curve 1 corresponds to the stochastic oscillations of the boundary and curve 3 corresponds to its regular perturbations. Approximations of the experimental dependencies by the function $y \sim an^c$, where a and c are the fitting parameters, are shown as full lines. For the stochastic case these parameters are: $a_s = 0.0129 \pm 0.0001$, $c_s = 0.495\,46 \pm 0.000\,06$. For regular perturbations of the boundary $a_r = 0.0082 \pm 0.0001$, $c_r = 0.5445 \pm 0.0001$. One can see that experimental results and approximating curves are almost coincident with each other. In addition, both curves are well described by the square-root dependence. For comparison, in figure 2 the analytically obtained dependence (14) is shown (broken curve 2). The difference between curves 1 and 2 can be explained by the assumption that $v \gg u_0$, which we used in the derivation of the dependence (14).

Thus, in the case of regular perturbations of the boundary the average velocity of the billiard particles increases more rapidly than for the stochastic perturbations.

Figure 3 shows the dependencies of particle velocities on time. Dynamics of the particle ensemble was simulated up to 1.5×10^5 time units, and some trajectories (of the particles with

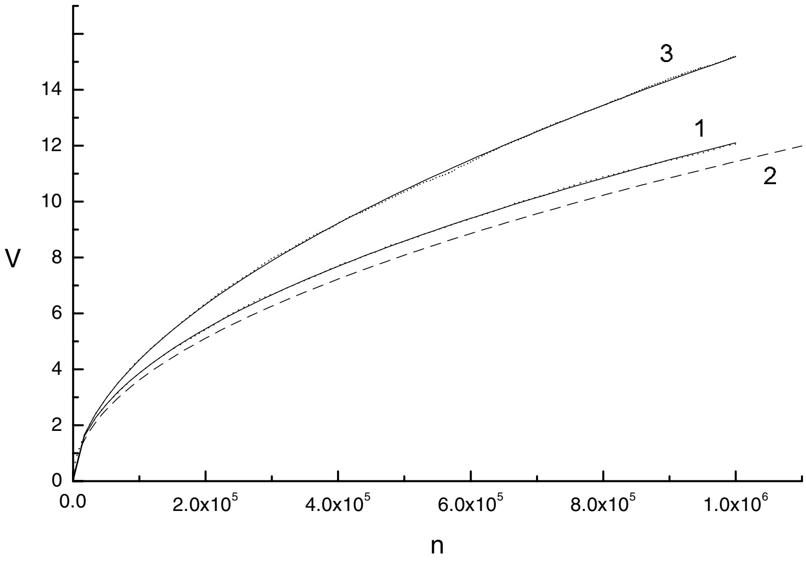

Figure 2. Average particle velocities as functions of the number n of collisions during 10^6 iterations (dotted curves) in the Lorentz gas. The following parameters were used: $u_0 = 0.01$, $a = 1$, $R = 0.4$, $\omega = 1$, and $v_0 = 1$. Curve 1 corresponds to the stochastic oscillations of the boundary and curve 3 corresponds to its regular perturbations. The average curve was obtained on the basis of 5000 realizations of different directions of the initial velocity v_0, which were selected in a random way. Approximations of the experimental dependencies by the function $y \sim an^c$ are shown as full lines. For the stochastic case the fitting parameters are: $a_s = 0.0129 \pm 0.0001$, $c_s = 0.495\,46 \pm 0.000\,06$. For regular perturbations of the boundary $a_r = 0.0082 \pm 0.0001$, $c_r = 0.5445 \pm 0.0001$. The analytically obtained dependence (14) is shown as the broken curve 2.

high velocities) include up to 10^7 iterations. Curve 1 corresponds to the stochastic perturbation; curve 2 is the result of the regular oscillations of the boundaries. Approximations of the numerical results by the dependence $y \sim at^c$ are shown as full lines. As in the previous case, the experimental and fitting curves are almost merged. For the stochastic perturbations the fitting parameters are the following: $a_s = (6.3 \pm 0.4) \times 10^{-5}$, $c_s = 1.0 \pm 0.0005$. For the regular oscillations of the boundary $a_r = (6.5 \pm 0.1) \times 10^{-6}$, $c_r = 1.236 \pm 0.002$. The difference between values a_s and a_r arises due to various exponents c. For comparison, in figure 3, the analytical curve (16) is shown (broken line 2). One can see that, in the stochastic case, the average velocity increases as a linear function of time. It is in good agreement with equation (16). In the regular case the particle acceleration is larger, and $V \sim t^c$, where $c > 1$.

As follows from analytical estimations of the dispersion (see section 3.2) and detailed numerical results(see [25]), the values $\overline{\Delta v}$ and $\overline{\Delta v^2}$ are larger for the case of regular oscillations of the boundary. So, the particle acceleration is also larger. Furthermore, for this case $\overline{\Delta v}$ and $\overline{\Delta v^2}$ are the functions of the particle velocity. Therefore, the velocity distribution function differs from the normal distribution (12), and the dependence of the average velocity is not linear with time t.

4. Stadium with time-dependent boundary: numerical investigations

Stadium-like billiards are defined as a closed domain Q with the boundary ∂Q consisting of two parallel lines and two focusing curves, i.e. $\partial Q = \partial Q^- \bigcup \partial Q^0$ (figure 4). The mechanism

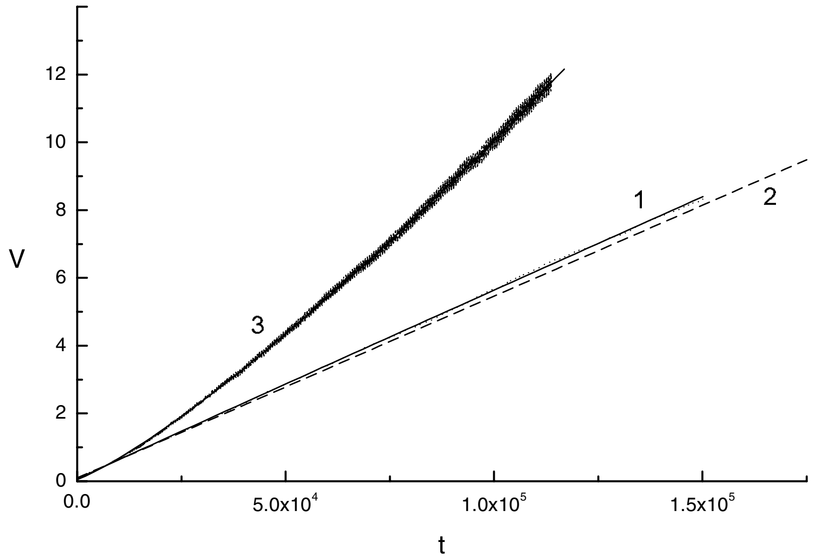

Figure 3. The same data as in figure 2 but for dependence on time. The fitting parameters are the following: $a_s = (6.3 \pm 0.4) \times 10^{-5}$, $c_s = 1.0 \pm 0.0005$ (stochastic oscillations of the boundary) and $a_r = (6.5 \pm 0.1) \times 10^{-6}$, $c_r = 1.236 \pm 0.002$. The dependence (16) is shown as the broken curve 2.

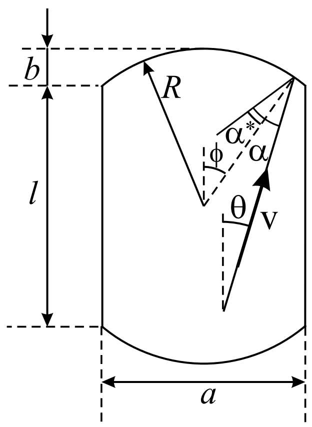

Figure 4. Billiard in a stadium and its dynamical variables.

of chaos in such billiards has been described in [2, 4, 5] (see also references therein). This mechanism is said to be a defocusing one and consists of the fact that, after reflection, the narrow beam of trajectories is defocused before the next reflection. In addition, along the trajectory the defocusing period should be longer than the focusing one on average.

To describe the particle dynamics in billiards with neutral components the known method of specular reflections is often used (see, e.g., [16]). In the given case it can be introduced as follows: the billiard domain Q is reflected specularly with respect to one of the neutral components ∂Q^0. As a result we will have collisions of the billiard particle only with ∂Q^-.

Suppose that focusing components ∂Q^- of the billiard are arcs of a circle of a radius R and an angle measure 2Ψ. Then $R = (a^2+4b^2)/8b$ and $\Psi = \arcsin[a/(2R)]$ (figure 4). Let us take that the components $\partial Q^- = \partial Q^-(t)$ oscillate in the following way: $R = R_0 + rf(t)$, where $r \ll R_0$ and $f(t)$ is a harmonic function. Therefore the boundary velocity is $u = u_0 \dot{f}(t)$, where $u_0 = r\omega$ is an amplitude and ω is a frequency of oscillation. In contrast to the time-dependent Lorentz gas (see section 2), the billiard map in the stadium of such a configuration should take into account multiple collisions with focusing components.

Let us introduce dynamical variables as shown in figure 4. Obviously, $\theta, \alpha, \alpha^* \in [-\pi/2, \pi/2]$ and $\phi \in [-\Psi, \Psi]$. Suppose that t_n is a collision time and v_n is a particle velocity just before the nth collision. Then the billiard map can be written as follows:

$$\begin{array}{l}
v_{n+1} = \sqrt{v_n^2 - 4v_n u_n \cos\alpha_n + 4u_n^2} \\
\alpha^*_{n+1} = \arcsin\left(\dfrac{v_n}{v_{n+1}} \sin\alpha_n\right) \\
\left.\begin{array}{l}
\alpha_{n+1} = \alpha^*_n \\
\phi_{n+1} = \phi_n + (\pi - 2\alpha^*_n) \pmod{2\pi} \\
t_{n+1} = t_n + \frac{2R\cos\alpha^*_n}{v_{n+1}}
\end{array}\right\} \quad \text{if} \quad |\phi_{n+1}| \leqslant \Psi \\
\left.\begin{array}{l}
\theta_{n+1} = \alpha^*_n - \phi_n \\
x^*_{n+1} = \frac{R}{\cos\theta_{n+1}}[\sin\alpha^*_n + \sin(\Psi - \theta_{n+1})] \\
x_{n+1} = x^*_{n+1} + l\tan\theta_{n+1} \pmod{a} \\
\alpha_{n+1} = \arcsin\left[\sin(\theta_{n+1} + \Psi) - \frac{x_{n+1}}{R}\cos\theta_{n+1}\right] \\
\phi_{n+1} = \theta_{n+1} - \alpha_{n+1} \\
t_{n+1} = t_n + \frac{R(\cos\phi_n + \cos\phi_{n+1} - 2\cos\Psi) + l}{v_{n+1}\cos\theta_{n+1}}
\end{array}\right\} \quad \text{if} \quad |\phi_n + \pi - 2\alpha^*_n| > \Psi
\end{array} \qquad (21)$$

where $u_n = u_0 \sin\omega t_n$. In the given map two auxiliary variables x and x^* are used. They define coordinates of the intersection point of the particle trajectory with the span connecting the end of arcs ∂Q^- before and after collisions, respectively. The first group in (21) corresponds to sequential multiple collisions with one of the focusing components, and the second group corresponds to the passage to the opposite side of the boundary.

Numerical investigations of the map (21) have been performed in two cases: when the billiard has strong chaotic properties and for a near-rectangle stadium. In the first case the billiard is a 'classical' stadium. Then $\Psi = \pi/2$ and the billiard is a domain with a boundary that consists of two semicircles and two parallel segments tangential to them. The latter case means that focusing components are segments of the almost straight line, and the billiard system is a near-integrable one.

For the first case the following billiard parameters were chosen: $a = 0.5$, $b = 0.25$, $l = 1$, $u_0 = 0.01$, $\omega = 1$ and $v_0 = 0.1$. The particle velocity was calculated as the averaged value of the ensemble of 5000 trajectories with different initial conditions (full curve 1 in figure 5). These initial conditions were different from each other by a random choice of the direction of the velocity vector v_0. As follows from the numerical analysis, the obtained dependence has approximately the same square-root behaviour as in the Lorentz gas ($V(n) \sim \sqrt{n}$). The fitting function $y \sim an^c$ (the chain curve 1 in figure 5) yields the following values: $a = 0.010\,15 \pm 0.000\,02$ and $c = 0.4446 \pm 0.0002$.

A near-integrable case means that the parameter b (see figure 4) is sufficiently small, and the curvature of the focusing components gives rise only to weak nonlinearity in the system. In such a configuration the billiard phase space has regions with regular and chaotic dynamics. This case is much more interesting for investigation.

As is known, the unperturbed billiard (the usual stadium, figure 4) has stable fixed points surrounded by invariant curves. In their neighbourhood the particle motion is quasiperiodic, and it can be well approximated by a twist map. At the same time, outside of these regions

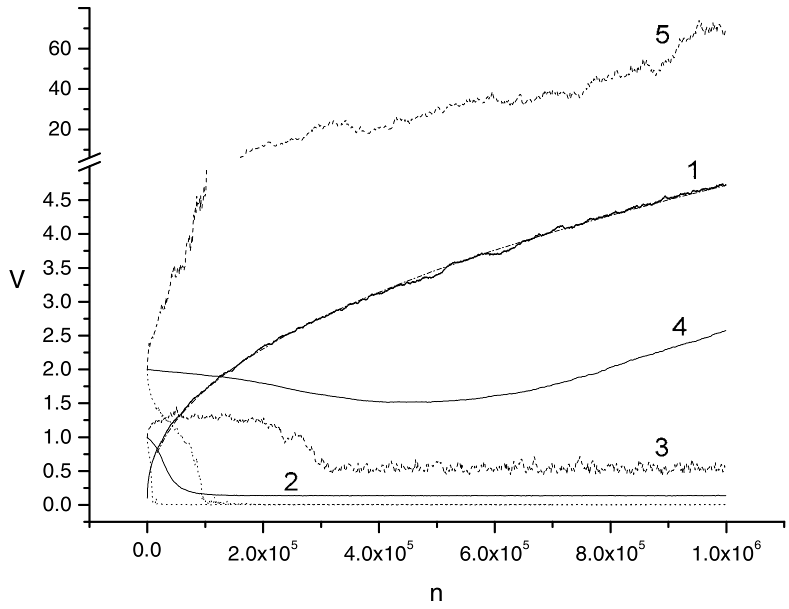

Figure 5. Average velocity of the ensemble of 5000 particles in a stadium as a function of the number of collisions, $l = 1$, $a = 0.5$, $u_0 = 0.01$ and $\omega = 1$. Two (chain and full) curves 1 correspond to the billiard with strong chaotic properties ($b = 0.25$). Approximation of the experimental dependencies (chain curve 1) by the function $y \sim an^c$ is shown as a full curve. The fitting parameters are: $a = 0.010\,15 \pm 0.000\,02$ and $c = 0.4446 \pm 0.0002$. Curves 2–5 correspond to the near-integrable system ($b = 0.005$): $v_0 = 1$ (curve 2, 3) and $v_0 = 2$ (curves 4, 5). Curves 2 and 4 are the average velocities of the particle ensemble. Curves 3 and 5 correspond to maximal velocities reached by the particle ensemble to the nth iteration.

the dynamics is chaotic. For this case the trajectory fills the whole chaotic region. Thus, if the boundary of the billiard is not perturbed then, depending on the initial conditions, the motion of billiard particles can be regular or chaotic.

Consider now the near-rectangle billiard with a time-dependent boundary. In such a billiard the particle can move from the chaotic region to the regular one and back. In a sufficiently small neighbourhood of stable fixed points the behaviour of the billiard particle has a certain rotation period, $T_1 \simeq 2\pi l/(wv)$, where w is a rotation number, l is the free path and v is the particle velocity. At the same time, the period of boundary oscillations is $T_2 = 2\pi/\omega$. Thus, in this system for some critical velocity v_c the resonance can be observed. As follows from numerical investigations, on each side of the resonance the behaviour of the particle velocity is essentially different. If the initial value $v_0 < v_c$ then the particle velocity decreases up to a certain quantity $v_{\rm fin} < v_c$ and the particle distribution tends to the stationary one in the interval $(0, v_{\rm fin})$. If, however, $v_0 > v_c$ then billiard particles can reach high velocities. In this case the particle distribution is *not* stationary, and it grows infinitely. In addition, the average particle velocity is also not bounded.

Similar to the billiard with fixed boundary, in a small enough neighbourhood of stable fixed points the regular motion is observed, but at $v \to v_c$ its size tends to zero. Inside such a neighbourhood (in the case of a sufficiently small perturbation), the particle velocity does not grow.

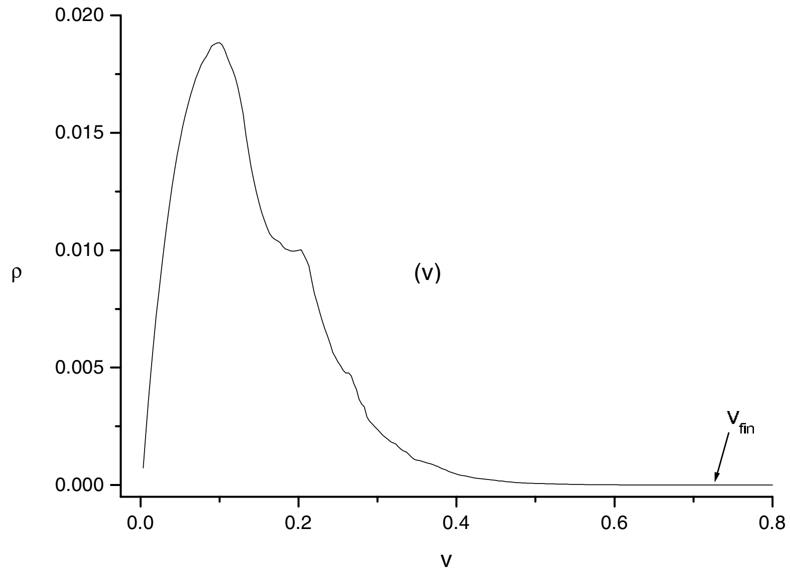

Figure 6. Stationary distribution of the particle velocity calculated by the one-particle trajectory during 10^9 iterations.

Thus, in the perturbed billiard we observe two different phenomena.

- First, if $v_0 > v_c$ then the averaged particle velocity increases.
- Second, at $v_0 < v_c$ the velocity distribution is stabilized and the particle velocity decreases up to a certain value v_{fin}.

For detailed numerical investigation initial conditions were randomly chosen in the chaotic region of the unperturbed billiard. In figure 5 the particle velocity as a function of the number of iterations is shown (curves 2–5). The billiard parameters remain the same as for the stadium-like billiard (curve 1), except for $b = 0.005$.

On the basis of 5000 realizations and for every initial velocity, *three curves* have been constructed: the average, minimal and maximal velocities which the particle ensemble has reached to the nth iteration. So, we can see the interval of the velocity change. As follows from this figure, if $v_0 < v_c$ then the averaged particle velocity (full curve 2) gradually decreases and tends to a constant. The maximal velocity of particles (dotted curve 3 in figure 5) also decreases up to v_{fin} and then fluctuates near this value. Eventually, the particle velocities lie in the interval $0 < v < v_{\text{fin}}$. In the case of $v > v_c$, the minimal velocity of particles decreases as before. This means that, in the ensemble, there is a number of particles which are in the region of low velocity values. In our numerical analysis the proportion of such particles was about 75%. At the same time, there are particles with high velocities (broken curve 5, which corresponds to the maximal velocity of the ensemble). As a result, the averaged particle velocity (full curve 4) increases.

In figure 6 a stationary velocity distribution is shown. This distribution was calculated by the one-particle trajectory during 10^9 iterations. The initial velocity was chosen as follows: $v_0 \approx v_{\text{fin}}/2$. The value denoted by v_{fin} corresponds to the maximally reached velocity.

Thus, numerically, in the near-rectangle stadium-like billiard with time-dependent boundary quite new and interesting phenomena are observed. The mechanism of particle separation by velocities requires detailed investigations that will be described in [30].

5. Conclusion

Billiards are very convenient models of several physical systems. For example, particle trajectories in billiards of specific configuration can be used in modelling a lot of dynamical systems. Moreover, most approaches to the problems of mixing in many-body systems go back to billiard-like questions. A natural physical generalization of a billiard system is a billiard whose boundary is not fixed, but varies by a certain law. This is quite a new field which opens up new prospects in studies of problems that have been known for a long time.

In this paper we have studied the problem of Fermi acceleration in dynamical systems generated by a two-dimensional Lorentz gas with time-dependent scatterer boundaries and the billiard in the form of a stadium with a periodically perturbed boundary. As is known, the usual Lorentz gas (with a fixed boundary) has strong chaotic properties (mixing, decay of correlations, etc.). Perturbations of the boundaries in such a billiard lead to the Fermi acceleration of the particle appearing. It is found that the acceleration is higher in the case of periodic boundary oscillations.

It is quite clear that the ideas described can be applied to an arbitrary billiard in which a distribution of angle α (between the normal to the boundary at a collision point and the particle velocity) is known. Therefore, the technique developed can be used in solving the problem of Fermi acceleration in a general case.

Numerical analysis of stadium-like billiards shows that, for the case of the developed chaos, the dependence of the particle velocity on the number of collisions has the same character as in the Lorentz gas. At the same time, for a near-rectangle stadium an interesting phenomena is observed. Depending on the initial values, the particle ensemble can be accelerated, or its velocity can decrease up to quite a low magnitude. However, if the initial values do not belong to a chaotic layer then for quite high velocities the particle acceleration is not observed.

Thus, on the basis of our investigations we can advance the following conjecture: chaotic dynamics of a billiard with a fixed boundary is a sufficient condition for the Fermi acceleration in the system when a boundary perturbation is introduced.

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